On Oseen's approximation

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An investigation is made into the validity of the Oseen equations, for incompressible, viscous flow past a body, as an approximation to the Navier–Stokes equations. It is shown that, when the body is such that a reversal of the uniform flow at infinity merely reverses any component of the force on the body without changing its absolute magnitude, that component can be determined correctly to the first order in the Reynolds number, though the detailed velocity field is not correct to this order. Moreover, this force can be deduced simply from a knowledge of the force on the body according to Stokes's approximation.

The analysis is also generalized to include the magneto-hydrodynamic effects when the fluid is conducting and the flow takes place in the presence of a magnetic field.

1. Introduction

Since the appearance of Stokes's approximate solution for the flow of a viscous fluid past a sphere (Stokes 1851), numerous attempts have been made, both to generalize the problem by changing the shape of the body, and to improve the calculation by including the effect of the inertia terms which were neglected in the original calculation. Oseen (1927), in particular, studied the problem extensively. Using the approximation which now bears his name, he gave solutions for the flow past various bodies at small Reynolds number (R) and calculated the force to the first order in R, one term more than would be given by the Stokes approximation. It is, however, generally recognized that the two approximations are of comparable accuracy, at least in the vicinity of the body. By the inclusion of the effect of the inertia terms, Oseen improved the flow picture far from the body where the Stokes approximation is inadequate, but near the body the difference between the two solutions is of an order of smallness which is outside the accuracy of either approximation. Oseen's calculation for the force thus requires some further justification, and this has recently been supplied, for flow past a sphere, by the work of Kaplun (1957) and Proudman & Pearson (1957). It appears that although Oseen failed to calculate correctly the velocity field, his result for the drag on the sphere, namely

$$D = D_S(1 + \tfrac{3}{8}R),$$

where D_S is the Stokes drag, is in fact valid because the correction to the velocity field makes no contribution to the total force on the sphere.

It is the purpose of this investigation to present a general criterion for the use of Oseen's equations as an approximate representation of the full Navier–Stokes equations. For the calculation of the force to the first order in the Reynolds number, it will be shown that the approximation is adequate provided that the force on the body is reversed in direction without change of magnitude, when the uniform flow at infinity is reversed. Furthermore, when this condition is satisfied, the force on the body can be deduced, correct to the first order in R, merely from a knowledge of the flow according to the Stokes approximation.

Because of the recent interest which has been shown in the corresponding magneto-hydrodynamic problem, the analysis has been generalized to include the effect of a magnetic field, aligned at infinity with the uniform stream, on the flow of a conducting fluid. Several special cases have been considered. On the assumption that the Reynolds number and magnetic Reynolds number were negligibly small, Chester (1957) found the drag on a sphere to be

$$D = D_S(1+\tfrac{3}{8}M),$$

correct to the first order in the Hartmann number. Chang (1960) generalized this to include any body of revolution aligned with its axis parallel to the uniform stream at infinity. Gotoh (1960*a*, *b*) considered the flow past a sphere including the first-order effects of Reynolds number and magnetic Reynolds number by an Oseen-type approximation. These results are special cases of a general formula valid for any body which satisfies the criterion stated in the previous paragraph.

Brenner (1961) has anticipated the results for the solution of the classical Oseen equations, and claims greater generality for them than is demonstrated in the present paper. But his arguments, lacking what he calls 'formal details of our procedure' do not seem to me to be convincing.

2. Mathematical formulation of the problem

We consider the steady flow of an incompressible, viscous, conducting fluid past a three-dimensional body of finite size. At infinity the flow is uniform and parallel to the x-axis. A magnetic field is imposed which is also uniform and parallel to the x-axis at infinity.

The equations to be solved are then, with the usual notation for electromagnetic quantities (measured in M.K.S. units),

$$\left.\begin{aligned} &\nabla' \wedge \mathbf{H}' = \sigma(\mathbf{E}' + \mu\mathbf{V}' \wedge \mathbf{H}'), \quad \nabla' . \mathbf{H}' = 0, \quad \nabla' \wedge \mathbf{E}' = 0, \\ &\nabla' . \mathbf{V}' = 0, \quad \rho(\mathbf{V}' . \nabla')\mathbf{V}' = -\nabla' p' - \rho\nu\nabla' \wedge (\nabla' \wedge \mathbf{V}') + \mu(\nabla' \wedge \mathbf{H}') \wedge \mathbf{H}', \end{aligned}\right\} \quad (2.1)$$

where $\mathbf{V}', p', \rho, \nu$ denote respectively the velocity, pressure, density and kinematic viscosity. The prime has been used here so that the same symbols without the prime can later be used to denote non-dimensional quantities.

We assume that the body lies within a sphere of radius L and centre at the origin of co-ordinates. The speed of the uniform stream at infinity is denoted by U_∞, and the magnitude of the magnetic field at infinity is denoted by H_∞. The space co-ordinates can then be made non-dimensional with the factor L^{-1}, and the dependent variables as follows,

$$\mathbf{H}' = H_\infty \mathbf{H}, \quad \mathbf{E}' = \mu U_\infty H_\infty \mathbf{E}, \quad \mathbf{V}' = U_\infty \mathbf{V}, \quad p' - p'_\infty = \rho\nu U_\infty p/L. \quad (2.2)$$

Equations (2.1) then become

$$\left.\begin{aligned} &\nabla\wedge\mathbf{H} = R_m(\mathbf{E}+\mathbf{V}\wedge\mathbf{H}), \quad \nabla.\mathbf{H} = 0, \quad \nabla\wedge\mathbf{E} = 0, \\ &\nabla.\mathbf{V} = 0, \quad R(\mathbf{V}.\nabla)\,\mathbf{V} = -\nabla p - \nabla\wedge(\nabla\wedge\mathbf{V}) + M^2(\mathbf{E}+\mathbf{V}\wedge\mathbf{H})\wedge\mathbf{H}, \end{aligned}\right\} \tag{2.3}$$

where

$$\left.\begin{aligned} R &= U_\infty L/\nu = \text{Reynolds number}, \\ R_m &= U_\infty L\mu\sigma = \text{magnetic Reynolds number}, \\ M &= \mu H_\infty L(\sigma/\rho\nu)^{\frac{1}{2}} = \text{Hartmann number}, \end{aligned}\right\} \tag{2.4}$$

these three parameters being essentially non-negative.

In the general problem a solution of equations (2.3) is required subject to the relevant boundary conditions on the body and at infinity. In what follows the nature of this solution is discussed when the three parameters defined by (2.4) are all small compared with unity.

3. The method of solution

The arguments used are based on the theory developed by Kaplun (1957) and applied by Proudman & Pearson (1957). For a detailed account the reader is referred to the above papers; here the discussion is restricted to a less formal summary of those aspects which are directly relevant to the present problem.

The Kaplun theory was, in fact, developed to deal with the difficulties associated with the approximate solution of equations such as (2.3). It has long been recognized that a straightforward perturbation approach breaks down, the reason being that in different regions of the flow, different terms of the equations are dominant. For example, in the classical Stokes solution for flow past a sphere (with $R = R_m = M = 0$), the term $\nabla\wedge(\nabla\wedge\mathbf{V})$ in the equation of motion is $O(r^{-3})$, when the radial co-ordinate r is large. But this solution is based on the neglect of the inertia term $R(\mathbf{V}.\nabla)\,\mathbf{V}$ which is merely $O(Rr^{-2})$. Thus if Rr is of order one or larger, the Stokes solution must fail, and although it satisfies the boundary condition at infinity, it does not give a reliable picture of the flow at large distances. In any case the fact that it does satisfy the boundary condition is exceptional, one finds that a higher approximation which satisfies all the boundary conditions cannot be found.

On the other hand, the Oseen approximation, based on a perturbation about the uniform flow at infinity, does give a uniformly valid zero-order approximation. For this is a good approximation to the flow at large distances, and is no worse than the Stokes approximation near the body, at least to zero order in R. However such an approach does not improve the Stokes solution near the body; as far as higher approximations are concerned the boundary condition of zero velocity at the body is not consistent with linearization about a uniform stream. In short the Stokes solution is adequate for the crudest approximation to the flow near the body, but higher approximations require a more careful investigation of the flow at large distances, in order that the boundary conditions there shall be satisfied to the required accuracy.

To overcome the difficulty, two approximations are obtained, an inner solution which satisfies the boundary conditions on the body, and an outer solution which satisfies the boundary conditions at infinity. Thus neither solution is required to satisfy all the conditions, and to determine them completely it is assumed that both solutions are valid in some region where they overlap, and can hence be matched there to the appropriate degree of accuracy.

For the problem formulated in the previous paragraph, the inner solution will be of the form

$$\left.\begin{aligned} \mathbf{V} &= \mathbf{v}_0+\mathbf{v}_1+\dots, \\ p &= p_0+p_1+\dots, \\ \mathbf{H} &= \mathbf{h}_0+\mathbf{h}_1+\dots, \\ \mathbf{E} &= \mathbf{e}_0+\mathbf{e}_1+\dots. \end{aligned}\right\} \tag{3.1}$$

The leading terms, equivalent to the Stokes solution, satisfy the equations

$$\left.\begin{aligned} &\nabla\wedge\mathbf{h}_0 = 0, \quad \nabla.\mathbf{h}_0 = 0, \quad \nabla\wedge\mathbf{e}_0 = 0, \quad \nabla.(\mathbf{e}_0+\mathbf{v}_0\wedge\mathbf{h}_0) = 0, \\ &\nabla.\mathbf{v}_0 = 0, \quad -\nabla p_0-\nabla\wedge(\nabla\wedge\mathbf{v}_0) = 0. \end{aligned}\right\} \tag{3.2}$$

Note that the equation $\nabla.(\mathbf{e}_0+\mathbf{v}_0\wedge\mathbf{h}_0) = 0$, which is a consequence of the first of equations (2.3), is required to make the approximation consistent. It determines the field $\mathbf{e}_0$, which only appears when $\nabla.(\mathbf{v}_0\wedge\mathbf{h}_0)$ is different from zero.

The next approximation is obtained from (2.3) by retaining terms of the first order in R and R_m (there is no term of order M) and substituting therein the Stokes solution. The resulting equations are

$$\left.\begin{aligned} &\nabla\wedge\mathbf{h}_1 = R_m(\mathbf{e}_0+\mathbf{v}_0\wedge\mathbf{h}_0), \quad \nabla.\mathbf{h}_1 = 0, \quad \nabla\wedge\mathbf{e}_1 = 0, \quad \nabla.(\mathbf{e}_1+\mathbf{v}_1\wedge\mathbf{h}_0+\mathbf{v}_0\wedge\mathbf{h}_1) = 0, \\ &\nabla.\mathbf{v}_1 = 0, \quad R(\mathbf{v}_0.\nabla)\,\mathbf{v}_0 = -\nabla p_1-\nabla\wedge(\nabla\wedge\mathbf{v}_1). \end{aligned}\right\} \tag{3.3}$$

Note that the equations for $\mathbf{v}_0$ and $\mathbf{v}_1$ are independent of the electromagnetic field, and although the latter field, together with its continuation inside the body, must satisfy the usual continuity conditions at the surface of the body, it is not necessary to consider this explicitly as far as the volocity is concerned.

The Stokes solution has the following property; if the uniform streaming motion at infinity is reversed in direction, the appropriate solution is

$$-\mathbf{v}_0, \quad -p_0, \quad \mathbf{h}_0, \quad -\mathbf{e}_0.$$

In general, the next approximation does not have this simple property, and if we write $\mathbf{v}_1 = \mathbf{v}_\sigma+\mathbf{v}_\alpha, p_1 = p_\sigma+p_\alpha$ for the original solution, and $(-\mathbf{v}_\sigma+\mathbf{v}_\alpha), (-p_\sigma+p_\alpha)$ when the flow at infinity is reversed, it follows from (3.3) that

$$\left.\begin{aligned} &\nabla.\mathbf{v}_\alpha = 0, \quad R(\mathbf{v}_0.\nabla)\,\mathbf{v}_0 = -\nabla p_\alpha-\nabla\wedge(\nabla\wedge\mathbf{v}_\alpha), \\ &\nabla.\mathbf{v}_\sigma = 0, \qquad\qquad\qquad 0 = -\nabla p_\sigma-\nabla\wedge(\nabla\wedge\mathbf{v}_\sigma). \end{aligned}\right\} \tag{3.4}$$

In other words, $\mathbf{v}_\sigma$ is also a solution of Stokes's equations, though $\mathbf{v}_\alpha$ is not. The result will be required in the subsequent analysis.

Consider now the outer solution. The leading terms are a uniform streaming motion and a uniform magnetic field; the next approximation is obtained from a perturbation of this solution giving

$$\left.\begin{aligned} \mathbf{V} &= \mathbf{i}+\mathbf{V}_1+\dots, \\ p &= \quad P_1+\dots, \\ \mathbf{H} &= \mathbf{i}+\mathbf{H}_1+\dots, \\ \mathbf{E} &= \quad \mathbf{E}_1+\dots, \end{aligned}\right\} \tag{3.5}$$

where $\mathbf{i}$ is a unit vector parallel to the x-axis and

$$\left.\begin{aligned} &\nabla\wedge\mathbf{H}_1 = R_m(\mathbf{E}_1+\mathbf{V}_1\wedge\mathbf{i}+\mathbf{i}\wedge\mathbf{H}_1), \quad \nabla.\mathbf{H}_1 = 0, \quad \nabla\wedge\mathbf{E}_1 = 0, \\ &\nabla.\mathbf{V}_1 = 0, \quad R\,\partial\mathbf{V}_1/\partial x = -\nabla P_1 - \nabla\wedge(\nabla\wedge\mathbf{V}_1) + M^2(\mathbf{E}_1+\mathbf{V}_1\wedge\mathbf{i}+\mathbf{i}\wedge\mathbf{H}_1)\wedge\mathbf{i}. \end{aligned}\right\} \quad (3.6)$$

As already stated, the inner and outer solutions are finally determined by matching to an appropriate degree of accuracy in a common region of overlap. For our purpose it is sufficient to compare the inner solution with the approximation to the outer solution to the first order in R, R_m and M. It will be shown that this approximation takes the form

$$\mathbf{V} = \mathbf{V}_0+\mathbf{V}_\sigma+\mathbf{V}_\alpha, \quad (3.7)$$

where $\mathbf{V}_0$, $\mathbf{V}_\sigma$ and $\mathbf{V}_\alpha$ are to be matched respectively with $\mathbf{v}_0$, $\mathbf{v}_\sigma$ and $\mathbf{v}_\alpha$. The matching of $\mathbf{V}_0$ and $\mathbf{v}_0$ gives the usual Stokes inner solution. The expression for $\mathbf{V}_\sigma$ is particularly simple. Explicitly

$$\mathbf{V}_\sigma = (\gamma/16\pi\rho\nu L U_\infty)\{F_{0x}\mathbf{i}+\tfrac{3}{2}F_{0y}\mathbf{j}+\tfrac{3}{2}F_{0z}\mathbf{k}\}, \quad (3.8)$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors parallel to the co-ordinate axes, $\mathbf{F}_0$ is the force on the body according to Stokes approximation, and

$$\begin{aligned} \gamma &= \frac{R^2-RR_m+2M^2}{\{(R-R_m)^2+4M^2\}^{\frac{1}{2}}}, \quad && M^2 > RR_m \\ &= R, && M^2 < RR_m. \end{aligned} \quad (3.9)$$

Thus $\mathbf{v}_\sigma$ is a solution of Stokes's equations which tends to a uniform flow at large distances from the body. If the Stokes solutions which tend to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively are known, say $\mathbf{v}_0, \mathbf{v}_0', \mathbf{v}_0''$, then $\mathbf{v}_\sigma$ is given by

$$\mathbf{v}_\sigma = (\gamma/16\pi\rho\nu L U_\infty)\{F_{0x}\mathbf{v}_0+\tfrac{3}{2}F_{0y}\mathbf{v}_0'+\tfrac{3}{2}F_{0z}\mathbf{v}_0''\}. \quad (3.10)$$

There does not seem to be a simple result for $\mathbf{v}_\alpha$, and the complete velocity field cannot be given with such generality. If, however, the body is such that a reversal of the flow at infinity reverses the force on the body without change of magnitude, then there is clearly no net contribution to the force from the velocity field $\mathbf{v}_\alpha$. For such a body the force is given by

$$\mathbf{F} = \mathbf{F}_0+(\gamma/16\pi\rho\nu L U_\infty)\{F_{0x}\mathbf{F}_0+\tfrac{3}{2}F_{0y}F_0'+\tfrac{3}{2}F_{0z}F_0''\}, \quad (3.11)$$

where $\mathbf{F}_0$, $\mathbf{F}_0'$, $\mathbf{F}_0''$ are the forces on the body associated with the velocity fields $\mathbf{v}_0$, $\mathbf{v}_0'$, $\mathbf{v}_0''$ respectively.

A simple example is the drag on a sphere of radius L, first calculated by Gotoh (1960*a*),

$$D = 6\pi\rho\nu L U_\infty(1+\tfrac{3}{8}\gamma).$$

There are many other solutions in the literature, for example, the ellipsoid at incidence, (Oseen 1927), all of which are consistent with (3.11).

For bodies of more general shape, (3.11) is correct for any component of the force which merely changes its sign without change of magnitude in reverse flow. It can be used, for example, to determine the drag on a body which has fore-and-aft symmetry.

4. The details of the Oseen solution

To verify the results described above, the general solution of equation (3.6) is required. These equations imply that

$$\left.\begin{aligned} \nabla.\mathbf{H}_1 = 0, \quad \nabla.\mathbf{V}_1 = 0, \quad -\nabla^2\mathbf{H}_1 = R_m\partial(\mathbf{V}_1-\mathbf{H}_1)/\partial x, \\ R\frac{\partial \mathbf{V}_1}{\partial x} = -\nabla\left(P_1+\frac{M^2}{R_m}\mathbf{H}_1.\mathbf{i}\right)+\nabla^2\mathbf{V}_1+\frac{M^2}{R_m}\frac{\partial \mathbf{H}_1}{\partial x}. \end{aligned}\right\} \tag{4.1}$$

A particular consequence of (4.1) is that

$$\nabla^2\{P_1+(M^2/R_m)\,\mathbf{H}_1.\mathbf{i}\} = 0. \tag{4.2}$$

Thus we put
$$P_1+(M^2/R_m)\,\mathbf{H}_1.\mathbf{i} = \partial\psi/\partial x, \tag{4.3}$$
where
$$\nabla^2\psi = 0. \tag{4.4}$$

We may then deduce from (4.1) that

$$\nabla^2\{R_m\mathbf{V}_1+\beta\mathbf{H}_1\} = \partial\{R_m(R-\beta)\,\mathbf{V}_1+(\beta R_m-M^2)\,\mathbf{H}_1+R_m\nabla\psi\}/\partial x \tag{4.5}$$

for any constant β. If in particular

$$\beta = (\beta R_m-M^2)/(R-\beta), \tag{4.6}$$

which gives the two possible values

$$\beta_1 = \alpha_1-R_m, \quad \beta_2 = -(\alpha_2+R_m), \tag{4.7}$$

where
$$\left.\begin{aligned} 2\alpha_1 = \{(R-R_m)^2+4M^2\}^{\frac{1}{2}}+(R+R_m), \\ 2\alpha_2 = \{(R-R_m)^2+4M^2\}^{\frac{1}{2}}-(R+R_m), \end{aligned}\right\} \tag{4.8}$$

then (4.1) and (4.5) imply that

$$\nabla^2\mathbf{S}_1 = \alpha_1\,\partial\mathbf{S}_1/\partial x, \quad \nabla^2\mathbf{S}_2 = -\alpha_2\,\partial\mathbf{S}_2/\partial x, \quad \nabla.\mathbf{S}_1 = 0, \quad \nabla.\mathbf{S}_2 = 0, \tag{4.9}$$

where
$$\left.\begin{aligned} R_m\mathbf{S}_1 = R_m\mathbf{V}_1-(\alpha_2+R_m)\,\mathbf{H}_1+(R_m/\alpha_1)\,\nabla\psi, \\ R_m\mathbf{S}_2 = R_m\mathbf{V}_1+(\alpha_1-R_m)\,\mathbf{H}_1-(R_m/\alpha_2)\,\nabla\psi, \end{aligned}\right\} \tag{4.10}$$

or
$$\mathbf{H}_1 = \frac{R_m}{\alpha_1+\alpha_2}(\mathbf{S}_2-\mathbf{S}_1)+\frac{R_m}{\alpha_1\alpha_2}\nabla\psi, \quad \mathbf{V}_1 = \mathbf{H}_1+\frac{\alpha_1\mathbf{S}_1+\alpha_2\mathbf{S}_2}{\alpha_1+\alpha_2}. \tag{4.11}$$

The general solutions of equations (4.9) are of the form

$$\mathbf{S}_1 = e^{\alpha_1 x}\{\nabla\phi_1+\nabla\chi_1\wedge\mathbf{i}\}, \quad \mathbf{S}_2 = e^{-\alpha_2 x}\{\nabla\phi_2+\nabla\chi_2\wedge\mathbf{i}\}, \tag{4.12}$$

where
$$(\nabla^2+\alpha_1\,\partial/\partial x)\,\phi_1,\chi_1 = 0, \quad (\nabla^2-\alpha_2\,\partial/\partial x)\,\phi_2,\chi_2 = 0. \tag{4.13}$$

For $e^{\alpha_1 x}\nabla\phi_1$ is certainly a possible solution for $\mathbf{S}_1$, and this may be taken to include the most general expression for the x-component of $\mathbf{S}_1$. The most general solution is then obtained by adding a vector with zero x-component, and which satisfies the appropriate equations (4.9). The continuity equation implies that such a vector is of the form $\nabla\chi_1\wedge\mathbf{i}$, and the equation satisfied by χ_1 then follows from the first of equations (4.9).

The form of the expansions for $\phi_1, \phi_2, \chi_1, \chi_2$ which satisfy (4.13) will depend on the signs of α_1 and α_2. It is clear from (4.8) that α_1 cannot be negative, though α_2 may be positive or negative. We assume first that both α_1 and α_2 are positive.

It is convenient to begin with the expansion for ψ. By (4.4) this is a harmonic function, and (4.3) implies that $\partial\psi/\partial x$ is regular outside the body since both P_1 and $\mathbf{H}_1.\mathbf{i}$ must be regular. It is not, however, possible to deduce that ψ itself is regular. It was first shown by Goldstein (1931) that such an assumption is too restrictive as far as the general solution of the classical Oseen equation is concerned, and the arguments used by Goldstein can be extended to deal with the present situation.

Since $\partial\psi/\partial x$ is a regular harmonic function outside the body, the general expression for this derivative may be written in the form

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\rho^n(L_{np}\cos n\theta+L'_{np}\sin n\theta)\left(\frac{\partial}{\partial x}\right)^p\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\left(\frac{1}{r}\right). \tag{4.14}$$

Here (x,ρ,θ) are cylindrical polar co-ordinates, and r is the radial co-ordinate measuring the distance from the origin. Hence r^{-1} represents the fundamental solution of Laplace's equation which is regular at infinity. The rational harmonics of negative degree can all be represented as linear combinations of the terms in (4.14). This representation is used, rather than the more conventional expression involving orthogonal functions, because of its advantage in the subsequent development of the complete solution.

A possible expression for ψ, having an x-derivative of the form (4.14), is

$$\begin{aligned}\psi=&-\sum_{n=0}^{\infty}\rho^n(K_n\cos n\theta+K'_n\sin n\theta)\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\log(r-x)\\&+\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\rho^n(K_{np}\cos n\theta+K'_{np}\sin n\theta)\left(\frac{\partial}{\partial x}\right)^p\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\left(\frac{1}{r}\right).\end{aligned} \tag{4.15}$$

Although the first series is not regular along the positive x-axis, there is no reason to reject it unless the behaviour persists in the expressions for the physical variables. By (4.11), the contributions from ψ to the x-components of both $\mathbf{V}_1$ and $\mathbf{H}_1$ are regular, hence the same must be true of the contributions from $\mathbf{S}_1$ and $\mathbf{S}_2$. It follows from (4.12) that both $\partial\phi_1/\partial x$ and $\partial\phi_2/\partial x$ must be regular. Now regular solutions of the two equations (4.13) can be written in the form (4.14) with r^{-1} replaced by $e^{-\frac{1}{2}\alpha_1(r+x)}/r$ and $e^{-\frac{1}{2}\alpha_2(r-x)}/r$ respectively. It follows that general expressions for ϕ_1 and ϕ_2 may be written in the form

$$\begin{aligned}\phi_1=&-\sum_{n=0}^{\infty}\rho^n(A_n\cos n\theta+A'_n\sin n\theta)\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\int_{\frac{1}{2}\alpha_1(r+x)}^{\infty}\frac{e^{-s}}{s}ds\\&+\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\rho^n(A_{np}\cos n\theta+A'_{np}\sin n\theta)\left(\frac{\partial}{\partial x}\right)^p\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\frac{\exp[-\frac{1}{2}\alpha_1(r+x)]}{r},\end{aligned} \tag{4.16}$$

$$\begin{aligned}\phi_2=&\sum_{n=0}^{\infty}\rho^n(B_n\cos n\theta+B'_n\sin n\theta)\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\int_{\frac{1}{2}\alpha_2(r-x)}^{\infty}\frac{e^{-s}}{s}ds\\&+\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\rho^n(B_{np}\cos n\theta+B'_{np}\sin n\theta)\left(\frac{\partial}{\partial x}\right)^p\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\frac{\exp[-\frac{1}{2}\alpha_2(r-x)]}{r}.\end{aligned} \tag{4.17}$$

Finally χ_1 and χ_2 must be expressed as regular solutions of (4.13) with the addition of appropriate singular solutions which render the remaining components of $\mathbf{V}_1$ and $\mathbf{H}_1$ regular. This is possible provided that $K_0=0$. It then follows that $A_0 = B_0 = 0$, for otherwise the relevant terms in ϕ_1 and ϕ_2 would imply an infinite flux across planes $x = \text{const.}$

It may be verified that appropriate expressions for χ_1 and χ_2 are

$$\begin{aligned}\chi_1 = {} & \frac{e^{-\alpha_1 x}}{\alpha_1}\sum_{n=1}^{\infty}\rho^n(K_n'\cos n\theta - K_n\sin n\theta)\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\int_{\frac{1}{2}\alpha_1(r-x)}^{\infty}\frac{e^{-s}}{s}ds \\ & + \sum_{n=1}^{\infty}\rho^n(A_n'\cos n\theta - A_n\sin n\theta)\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\int_{\frac{1}{2}\alpha_1(r+x)}^{\infty}\frac{e^{-s}}{s}ds \\ & + \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\rho^n(C_{np}\cos n\theta + C_{np}'\sin n\theta)\left(\frac{\partial}{\partial x}\right)^p\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\frac{\exp\left[-\frac{1}{2}\alpha_1(r+x)\right]}{r}, \end{aligned} \quad (4.18)$$

$$\begin{aligned}\chi_2 = {} & \frac{e^{\alpha_2 x}}{\alpha_2}\sum_{n=1}^{\infty}\rho^n(K_n'\cos n\theta - K_n\sin n\theta)\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\left[\int_{\frac{1}{2}\alpha_2(r+x)}^{\infty}\frac{e^{-s}}{s}ds + 2\log\rho\right] \\ & - \sum_{n=1}^{\infty}\rho^n(B_n'\cos n\theta - B_n\sin n\theta)\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\int_{\frac{1}{2}\alpha_2(r-x)}^{\infty}\frac{e^{-s}}{s}ds \\ & - \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\rho^n(D_{np}\cos n\theta + D_{np}'\sin n\theta)\left(\frac{\partial}{\partial x}\right)^p\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\frac{\exp\left[-\frac{1}{2}\alpha_2(r-x)\right]}{r}. \end{aligned} \quad (4.19)$$

In the strict application of the Kaplun theory, only the leading terms of the above expressions would be retained to give the outer solution, but the reasoning is perhaps easier to follow if the complete solution of Oseen's equations is retained. For it is clear that such a solution must reduce to a solution of Stokes's equations in the limit as R, R_m and M all tend to zero, and this implies certain conditions on the orders of magnitude of the coefficients appearing in (4.15)–(4.19). If P_1, $\mathbf{V}_1$ and $\mathbf{H}_1$ are all to be $O(1)$ in the limit as R, R_m and $M \to 0$, it follows that

$$\left.\begin{aligned} A &= b - a/\alpha_1, \quad B = b + a/\alpha_2, \quad C = c - \alpha_1 d, \\ D &= c + \alpha_2 d, \quad K = -a + (\alpha_1\alpha_2 k)/R_m, \end{aligned}\right\} \quad (4.20)$$

where a, b, c, d and k are all $O(1)$. Here A represents one of the coefficients A_n, A_{np}, A_n', A_{np}', and similarly for B, C, D, K. The detailed array of coefficients represented by a, b, c, d and k have suffices and primes appended to agree with the left-hand sides of equations (4.20).

The results so far obtained now enable approximate expressions to be calculated for the outer solution when R, R_m and M are small. The algebra is tedious but straightforward, and we simply quote the final expression for the velocity. When relations (4.15)–(4.20) are substituted in (4.11) and the appropriate approximations carried out, the result can be expressed in the form

$$\mathbf{i} + \mathbf{V}_1 = \mathbf{V}_0 + \mathbf{V}_\sigma + \mathbf{V}_\alpha + O(R^2 + R_m^2 + M^2). \quad (4.21)$$

The first term is $O(1)$. In it the coefficients a, b, c, d and k appear linearly, and R, R_m, M, α_1 and α_2 do not occur explicitly. The second and third terms are of the form

$$\begin{aligned}\mathbf{V}_\sigma &= -[\{\alpha_1^2 + \alpha_2^2 + R_m(\alpha_2 - \alpha_1)\}/(\alpha_1 + \alpha_2)]\{\tfrac{1}{4}a_{00}\mathbf{i} + \tfrac{3}{8}a_1\mathbf{j} + \tfrac{3}{8}a_1'\mathbf{k}\} \\ &= -\gamma\{\tfrac{1}{4}a_{00}\mathbf{i} + \tfrac{3}{8}\alpha_1\mathbf{j} + \tfrac{3}{8}a_1'\mathbf{k}\}, \end{aligned} \quad (4.22)$$

where

$$\gamma = (R^2 - RR_m + 2M^2)/\{(R - R_m)^2 + 4M^2\}^{\frac{1}{2}}; \quad (4.23)$$

$$\mathbf{V}_\alpha = (R + R_m)\,\overline{\mathbf{V}}_\alpha, \quad (4.24)$$

where $\bar{\mathbf{V}}_\alpha$ also depends linearly on the coefficients a, b, c, d and k but not explicitly on the parameters R, R_m, M, α_1 and α_2.

Matching between the inner and outer solution is now possible with the help of the approximate expression (4.21). One first obtains an inner solution $\mathbf{v}_0$ satisfying the boundary conditions on the body. This solution is to match at large distances from the body with the uniform field $\mathbf{i}$, which is the leading term of the outer solution, and gives the usual velocity field according to Stokes's approximation. The next term of the outer solution is then determined by choosing the coefficients in the general solution for $\mathbf{V}_1$ so that its approximation $\mathbf{V}_0$ matches with $\mathbf{v}_0$. Because the complete Oseen solution has been used in the previous analysis, $\mathbf{V}_0$ is itself a general solution of Stokes's equation, and the matching at this stage then consists of identifying $\mathbf{V}_0$ with $\mathbf{v}_0$. Thus when the Stokes flow $\mathbf{v}_0$ is known, it is possible to deduce the Oseen solution $\mathbf{V}_1$; for our purpose it is sufficient to note that the terms $\mathbf{V}_\sigma$ and $\mathbf{V}_\alpha$ are now given. The procedure continues by calculating the next term $\mathbf{v}_1$ of the inner solution so that, at large distances from the body, it matches with $(\mathbf{V}_\sigma+\mathbf{V}_\alpha)$. According to the argument which led to (3.4), $\mathbf{v}_1$ is itself regarded as the sum of two components, $\mathbf{v}_\sigma$ and $\mathbf{v}_\alpha$. That these components should match separately with the $\mathbf{V}_\sigma$ and $\mathbf{V}_\alpha$ appearing in (4.21) can be seen by consideration of the outer solution when the flow at infinity is reversed. If in such a flow we write

$$\left.\begin{aligned} \mathbf{V} &= -\mathbf{i}-\mathbf{V}_{1r}+\ldots, \\ p &= \quad\;\; -P_{1r}+\ldots, \\ \mathbf{H} &= \;\;\mathbf{i}+\mathbf{H}_{1r}+\ldots, \\ \mathbf{E} &= \quad\;\; -\mathbf{E}_{1r}+\ldots, \end{aligned}\right\} \tag{4.25}$$

by analogy with (3.5), then $\mathbf{V}_{1r}$, P_{1r}, $\mathbf{H}_{1r}$ and $\mathbf{E}_{1r}$ are seen to satisfy equations equivalent to (3.6) but with a formal change in the sign of R and R_m. By (4.8) this is equivalent to interchanging α_1 and α_2, and general solutions for the reversed flow field can be written down using these modifications in (4.11)–(4.20). But without further calculation, the approximation to the outer flow field, corresponding to (4.21), is now seen to be

$$-\mathbf{i}-\mathbf{V}_1 = -\mathbf{V}_0-\mathbf{V}_\sigma+\mathbf{V}_\alpha+O(R^2+R_m^2+M^2). \tag{4.26}$$

For the first term must differ from (4.21) only by a change of sign, since it reduces to Stokes's solution which has this property. This determines $\mathbf{V}_\sigma$ and $\mathbf{V}_\alpha$ and implies one change of sign in both since in (4.21) all three terms depend linearly on the coefficients a, b, c, d and k. One further change of sign in $\mathbf{V}_\alpha$ (but not in $\mathbf{V}_\sigma$) arises from the change in sign of the factor $(R+R_m)$.

An expression for $\mathbf{v}_\sigma$ can now be written down in terms of the velocity fields $\mathbf{v}_0$, $\mathbf{v}_0'$, $\mathbf{v}_0''$, which satisfy Stokes's equations and tend to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively at large distances from the body. The appropriate expression, which matches with (4.22) is clearly

$$\mathbf{v}_\sigma = -\gamma\{\tfrac{1}{4}a_{00}\mathbf{v}_0+\tfrac{3}{8}a_1\mathbf{v}_0'+\tfrac{3}{8}a_1'\mathbf{v}_0''\} \tag{4.27}$$

and the results anticipated in § 3 follow provided that a_{00}, a_1 and a_1' can be suitably related to the force on the body according to Stokes's approximate solution. It is shown in the appendix that, if $\mathbf{F}_0$ is this force, then

$$(a_{00}, a_1, a_1') = -(1/4\pi\rho\nu LU_\infty)(F_{0x}, F_{0y}, F_{0z}). \tag{4.28}$$

Thus, finally,

$$\mathbf{v}_\sigma = (\gamma/16\pi\rho\nu L U_\infty)\{F_{0x}\mathbf{v}_0 + \tfrac{3}{2}F_{0y}\mathbf{v}_0' + \tfrac{3}{2}F_{0z}\mathbf{v}_0''\}. \tag{4.29}$$

There remains the case $\alpha_2 < 0$. Here the general solutions for ϕ_2 and χ_2, as expressed in (4.17) and (4.19), are not valid and must be replaced by expansions of the form

$$\begin{aligned}\phi_2 = &-\sum_{n=1}^{\infty}\rho^n(B_n\cos n\theta + B_n'\sin n\theta)\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\int_{-\frac{1}{2}\alpha_2(r+x)}^{\infty}\frac{e^{-s}}{s}ds \\ &+\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\rho^n(B_{np}\cos n\theta + B_{np}'\sin n\theta)\left(\frac{\partial}{\partial x}\right)^p\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\frac{\exp[\frac{1}{2}\alpha_2(r+x)]}{r},\end{aligned} \tag{4.30}$$

$$\begin{aligned}\chi_2 = &-\frac{e^{\alpha_2 x}}{\alpha_2}\sum_{n=1}^{\infty}\rho^n(K_n'\cos n\theta - K_n\sin n\theta)\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\int_{-\frac{1}{2}\alpha_2(r-x)}^{\infty}\frac{e^{-s}}{s}ds \\ &+\sum_{n=1}^{\infty}\rho^n(B_n'\cos n\theta - B_n\sin n\theta)\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\int_{-\frac{1}{2}\alpha_2(r+x)}^{\infty}\frac{e^{-s}}{s}ds \\ &+\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\rho^n(D_{np}\cos n\theta + D_{np}'\sin n\theta)\left(\frac{\partial}{\partial x}\right)^p\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\frac{\exp[\frac{1}{2}\alpha_2(r+x)]}{r}.\end{aligned} \tag{4.31}$$

General expressions for ψ, ϕ_1 and χ_1 are given, as before, by (4.15), (4.16) and (4.18). The definitions of a, b, c, d and k, as given by (4.20), can be taken over without modification. The solutions for reversed flow are then obtained by a formal change in the sign of x throughout. The final result of the corresponding calculations is to give an approximate expression for the outer flow similar to (4.21) but with $\gamma = R$ in the definition of $\mathbf{V}_\sigma$, as reported in (3.9).

5. Comments

That the force on the body is indeed given by (3.11) can be verified directly by carrying through the analysis of the appendix in greater detail, but when $\mathbf{v}_\alpha$ is ignored the velocity field near the body is simply a modified Stokes solution, and the pressure and forces are clearly modified correspondingly. The Maxwell stress tensor does not explicitly contribute since, by (2.3), it is derived from a body force which is $O(M^2)$ near the body and so is not sensible to the first order in M.

The change in form of the relation for γ when $M^2 = RR_m$ corresponds to a qualitative change in the flow picture. When $\alpha_2 > 0$, the vorticity is exponentially small everywhere at large distances save in two parabolic regions where $\alpha_1(r-x)$ and $\alpha_2(r+x)$ are not large, these regions lying on either side of the body. But when $\alpha_2 < 0$, the corresponding regions are those for which $\alpha_1(r-x)$ and $-\alpha_2(r-x)$ are not large, both lying on the downstream side of the body. For the critical case $\alpha_2 = 0$, which includes the classical Oseen solution, there is only one such region. In this case ϕ_2 and χ_2 reduce to potential functions, in general having singularities along the x-axis. Physically speaking, the change-over marks the transition beyond which the disturbances associated with magneto-hydrodynamic effects cannot propagate upstream. With the Oseen approximation, the Alfven speed of such disturbances is $U_\infty M/(RR_m)^{\frac{1}{2}}$, and the qualitative picture of the flow changes abruptly according as this speed is greater or less than the convective speed (U_∞) of the fluid. The disappearance of appreciable vorticity in the parabolic region ahead of the body corresponds to a situation in which the disturbances associated with the magneto-hydrodynamic effects can no longer propagate upstream.

When R and R_m are negligible, $\gamma = M$ and certain conditions can be relaxed. There are no restrictions on the shape of the body, for $\mathbf{v}_\alpha \equiv 0$ and reversal of flow will always produce reversal of force, whatever the shape. The uniform flow at infinity need not be parallel to the magnetic field, since this property was only used in the Oseen-type approximation to include the effects of R and R_m. The use of inner and outer expansions is in fact not essential here. If the magnetic field is equal to $\mathbf{i}$ at infinity, then a composite solution will satisfy the equations

$$\left.\begin{aligned} \nabla.(\mathbf{E}+\mathbf{V}\wedge\mathbf{i}) = 0, \quad \nabla\wedge\mathbf{E} = 0, \\ \nabla.\mathbf{V} = 0, \quad 0 = -\nabla p - \nabla\wedge(\nabla\wedge\mathbf{V}) + M^2(\mathbf{E}+\mathbf{V}\wedge\mathbf{i}). \end{aligned}\right\} \tag{5.1}$$

For the magnetic field is adequately represented at large distances, and the fact that its representation may not be accurate near the body is of no consequence to the first order in M, since the body force is negligible to this order near the body. The convective terms, which are $O(R)$, are majorized uniformly by the body force when R is negligible, and so can be neglected even in the outer solution.

The analysis is easily adapted to the situation in which the uniform flow at infinity is in an arbitrary direction, and (3.11) requires only a slight modification to give the force. Let $\mathbf{F}_0$, $\mathbf{F}_0'$, $\mathbf{F}_0''$ be defined as in (3.11), and let $\mathbf{F}_S$ be the force on the body in the actual flow according to Stokes's approximation. Then

$$\mathbf{F} = \mathbf{F}_S + (M/16\pi\rho\nu LU_\infty)\{F_{sx}\mathbf{F}_0 + \tfrac{3}{2}F_{sy}\mathbf{F}_0' + \tfrac{3}{2}F_{sz}\mathbf{F}_0''\} + O(M^2). \tag{5.2}$$

For example, the drag on a sphere of radius L when moving parallel to the magnetic field is given by $6\pi\rho\nu LU_\infty(1+\tfrac{3}{8}M)$ (Chester 1957). When moving perpendicular to the magnetic field the drag is $6\pi\rho\nu LU_\infty(1+\tfrac{9}{16}M)$ (Gotoh 1960*b*).

This paper is an expanded version of a lecture to the British Theoretical Mechanics Colloquium in April 1961.

Appendix

With the help of the continuity equations for $\mathbf{V}'$ and $\mathbf{H}'$, the momentum equation in (2.1) may be written in the form $\operatorname{div}\boldsymbol{\tau}' = 0$, where $\boldsymbol{\tau}'$ is the symmetrical tensor given by

$$\tau'_{ij} = -p'\delta_{ij} + \rho\nu\left(\frac{\partial V'_i}{\partial x'_j} + \frac{\partial V'_j}{\partial x'_i}\right) - \tfrac{1}{2}\mu H'^2\delta_{ij} + \mu H'_i H'_j - \rho V'_i V'_j, \tag{A. 1}$$

and δ_{ij} is the unit tensor. Thus, by the divergence theorem,

$$\iint_{\text{Body}} \boldsymbol{\tau}'.\mathbf{n}\,dS' = \iint_{\Sigma'} \boldsymbol{\tau}'.\mathbf{n}\,dS', \tag{A. 2}$$

where Σ' is some surface surrounding the body, and $\mathbf{n}$ is the unit outward normal.

The integral over the body is just the force on the body. For the first two terms in $\boldsymbol{\tau}'$ are the usual contributions to the stress tensor, the terms involving $\mathbf{H}'$ come from the Maxwell stress tensor and the last term gives no contribution since $\mathbf{V}' = 0$ on the body. Equation (A. 2) is then equivalent to

$$\text{Force on body} = -\iint_{\Sigma'} [\mathbf{n}(p' + \tfrac{1}{2}\mu H'^2) + \rho\nu\mathbf{n}\wedge(\nabla'\wedge\mathbf{V}') - \mu(\mathbf{H}'.\mathbf{n})\,\mathbf{H}' + p(\mathbf{V}'.\mathbf{n})\,\mathbf{V}']\,dS'. \quad (A.3)$$

For the purpose of deriving (4.28) it is sufficient to calculate the force according to the first term of the inner solution. This can equally well be obtained from a calculation using the outer solution, by discarding all but the leading terms in the final expression for the force. Thus, to a sufficient order of accuracy,

$$\frac{\text{Force on body}}{\rho\nu LU_\infty} = -\iint_\Sigma \Big[\mathbf{n}\Big\{P_1 + \frac{M^2}{2R_m}(\mathbf{i}+\mathbf{H}_1).(\mathbf{i}+\mathbf{H}_1)\Big\} + \mathbf{n}\wedge(\nabla\wedge\mathbf{V}_1) - \frac{M^2}{R_m}\{(\mathbf{i}+\mathbf{H}_1).\mathbf{n}\}\{\mathbf{i}+\mathbf{H}_1\} + R\{(\mathbf{i}+\mathbf{V}_1).\mathbf{n}\}\{\mathbf{i}+\mathbf{V}_1\}\Big]\,dS \quad (A.4)$$

and the terms

$$-(M^2/R_m)\{(\mathbf{i}+\mathbf{H}_1).\mathbf{n}\}\,\mathbf{i} + R\{(\mathbf{i}+\mathbf{V}_1).\mathbf{n}\}$$

can be omitted from the integrand since they give zero contribution to the integral by continuity.

The surface Σ is chosen to be a large cylinder enclosing the body, whose ends lie in the planes S_+ and S_- defined by x = positive constant, x = negative constant respectively. Then the only sensible contributions to the integral, in the limit as the dimensions of Σ all tend to infinity, are from

$$-\Big\{\iint_{S+} dS - \iint_{S-} dS\Big\}\Big\{\mathbf{i}\Big(P_1 + \frac{M^2}{R_m}\mathbf{H}_1.\mathbf{i}\Big) + \mathbf{i}\wedge(\nabla\wedge\mathbf{V}_1) - \frac{M^2}{R_m}\mathbf{H}_1 + R\mathbf{V}_1\Big\} \quad (A.5)$$

$$= -\Big\{\iint_{S+} dS - \iint_{S-} dS\Big\}\Big\{-\mathbf{i}\wedge(\nabla\psi\wedge\mathbf{i}) + \frac{\alpha_1(\alpha_1 - R_m)}{\alpha_1+\alpha_2}e^{\alpha_1 x}\phi_{1x}\mathbf{i} - \frac{\alpha_2(\alpha_2+R_m)}{\alpha_1+\alpha_2}e^{-\alpha_2 x}\phi_{2x}\mathbf{i} + \frac{\alpha_1 - R_m}{\alpha_1+\alpha_2}e^{\alpha_1 x}\mathbf{i}\wedge\nabla\chi_{1x} + \frac{\alpha_2+R_m}{\alpha_1+\alpha_2}e^{-\alpha_2 x}\mathbf{i}\wedge\nabla\chi_{2x}\Big\}, \quad (A.6)$$

where equations (4.3), (4.11), (4.12) and (4.13) have been used.

The component in the x-direction arises from the terms in ϕ_1 and ϕ_2 which are independent of θ. With the help of (4.16) and (4.17) or (4.30) one gets for this component

$$-\Big\{\iint_{S+} dS - \iint_{S-} dS\Big\}\Big\{\frac{\alpha_1(\alpha_1-R_m)}{\alpha_1+a_2}\sum_{p=0}^{\infty} A_{0p}e^{\alpha_1 x}\Big(\frac{\partial}{\partial x}\Big)^{p+1}\frac{\exp[-\frac{1}{2}\alpha_1(r+x)]}{r} - \frac{\alpha_2(\alpha_2+R_m)}{\alpha_1+\alpha_2}\sum_{p=0}^{\infty} B_{0p}e^{-\alpha_2 x}\Big(\frac{\partial}{\partial x}\Big)^{p+1}\exp[\mp\tfrac{1}{2}\alpha_2(r\mp x)]\Big\}. \quad (A.7)$$

In the last term the upper sign is taken for $\alpha_2 > 0$, and the lower sign for $\alpha_2 < 0$.

The integrals can be evaluated with the help of the following result. For $\alpha > 0$

$$\iint_S \frac{e^{-\alpha r}}{r}\,dS = 2\pi\int_0^\infty \frac{\exp[-\alpha(x^2+\rho^2)^{\frac{1}{2}}]\,\rho\,d\rho}{(x^2+\rho^2)^{\frac{1}{2}}} = \frac{2\pi}{\alpha}e^{-\alpha|x|}. \quad (A.8)$$

Substitution in (A. 7) gives

$$-4\pi\left\{\frac{\alpha_1-R_m}{\alpha_1+\alpha_2}\sum_p A_{0p}\,e^{\alpha_1 x}\left(\frac{\partial}{\partial x}\right)^{p+1} e^{-\alpha_1 x}+\frac{\alpha_2+R_m}{\alpha_1+\alpha_2}\sum_p B_{0p}\,e^{-\alpha_2 x}\left(\frac{\partial}{\partial x}\right)^{p+1} e^{\alpha_2 x}\right\}$$

$$=-\frac{4\pi}{\alpha_1+\alpha_2}\sum_p\{(\alpha_1-R_m)\,A_{0p}(-\alpha_1)^{p+1}+(\alpha_2+R_m)\,B_{0p}\,\alpha_2^{p+1}\}. \quad \text{(A. 9)}$$

In the limit as R, R_m and $M\to 0$, this gives $-4\pi a_{00}$, as is readily inferred with the help of (4.20). Thus

$$F_{0x}/\rho\nu LU_\infty=-4\pi a_{00}. \quad \text{(A. 10)}$$

The contributions to the other components of (A. 6) arise only from those terms which have singularities on the x-axis. The y-component, for example, reduces to

$$-\left\{\iint_{S+}dS-\iint_{S-}dS\right\}\left\{\left(K_1\cos^2\theta\frac{\partial^2}{\partial\rho^2}+K_1\sin^2\theta\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\log(r-x)\right.$$

$$-\frac{\alpha_1-R_m}{\alpha_1+\alpha_2}\left(K_1\sin^2\theta\frac{\partial^2}{\partial\rho^2}+K_1\cos^2\theta\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\int_{\frac{1}{2}\alpha_1(r-x)}^{\infty}\frac{e^{-s}}{s}ds$$

$$\left.\pm\frac{\alpha_2+R_m}{\alpha_1+\alpha_2}\left(K_1\sin^2\theta\frac{\partial^2}{\partial\rho^2}+K_1\cos^2\theta\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\int_{\pm\frac{1}{2}\alpha_2(r\pm x)}^{\infty}\frac{e^{-s}}{s}dS\right\} \quad \text{(A. 11)}$$

$$=-\pi K_1\left\{\int_{0\,(x>0)}^{\infty}\rho\,d\rho-\int_{0\,(x<0)}^{\infty}\rho\,d\rho\right\}\left\{\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right\}$$

$$\times\left\{\log(r-x)-\frac{\alpha_1-R_m}{\alpha_1+\alpha_2}\int_{\frac{1}{2}\alpha_1(r-x)}^{\infty}\frac{e^{-s}}{s}ds\pm\frac{\alpha_2+R_m}{\alpha_1+\alpha_2}\int_{\pm\frac{1}{2}\alpha_2(r\pm x)}^{\infty}\frac{e^{-s}}{s}ds\right\} \quad \text{(A. 12)}$$

$$=4\pi K_1. \quad \text{(A. 13)}$$

By (4.20), the limiting value of K_1 is $-a_1$ and so

$$F_{0y}/\rho\nu LU_\infty=-4\pi a_1. \quad \text{(A. 14)}$$

Similarly, one can show that

$$F_{0z}/\rho\nu LU_\infty=-4\pi a_1' \quad \text{(A. 15)}$$

as required in (4.28).

REFERENCES

Brenner, H. 1961 *J. Fluid Mech.* **11**, 604.
Chang, I-Dee 1960 *J. Fluid Mech.* **9**, 473.
Chester, W. 1957 *J. Fluid Mech.* **3**, 304.
Goldstein, S. 1931 *Proc. Roy. Soc.* A, **131**, 198.
Gotoh, K. 1960*a* *J. Phys. Soc. Japan*, **15**, 189.
Gotoh, K. 1960*b* *J. Phys. Soc. Japan*, **15**, 696.
Kaplun, S. 1957 *J. Rat. Mech.* **6**, 585.
Oseen, C. W. 1927 *Neuere Methoden und Ergebnisse in der Hydrodynamik.* Leipzig Akademische Verlagsgesellschaft M.B.H.
Proudman, I. & Pearson, J. R. A. 1957 *J. Fluid Mech.* **2**, 237.
Stokes, G. G. 1851 *Camb. Phil. Trans.* **9**, 8.